Introduction to natural deduction

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1 Before starting...

This tutorial is available in several languages: Spanish[1] (and PDF), Esperanto[2] (and PDF), Catalan[3] (and PDF), and English[4] (and PDF).

Formulas look much nicer in the PDF, but if it's not possible to use it, then look at the HTML pages.

1.1 Who am I

My name is Daniel Clemente Laboreo, I'm 19 years old (in 2004), I live in Gavà (Barcelona, Spain), and I study Computer Science in the *FIB* (*UPC*, Public University of Catalonia). There, in the subject called *ILO* (*Introduction to logic*), is where I was taught this topic.

[1] http://www.danielclemente.com/logica/dn.html
[2] http://www.danielclemente.com/logica/dn.eo.html
[3] http://www.danielclemente.com/logica/dn.ca.html
[4] http://www.danielclemente.com/logica/dn.en.html

1.2 Why do I write this

Some reasons:

- There's a big gap in the search "*natural deduction*" at Google. I myself needed to study it before the exam, but couldn't find anything useful which helped me. There actually existed some tutorials, but no one was good enough (in my opinion): some were too confusing, others had special characters which didn't display correctly, and others didn't explain everything (as if everyone knew all the logic concepts). So I decided to create this tutorial which I hope it will help someone.
- It's a topic I like, and can do without much problems.
- It makes you think. Maybe it hasn't got a lot of practical uses, but one really has to try hard and spend some time in order to solve some simple problems.
- Well, I confess that I wrote this to learn text processing with LaTeX. You need some dedication to learn it, but the results make the work be worth it.

1.3 Whom is it addressed to

Principally, to anyone who likes logic, computer science, or mathematics. Anyone who wants to prepare the university logic subjects will also gain some useful concepts.

This doesn't pretend to be a complete course for natural deduction, but it will continue being an introduction. When I learn more, I will correct it if necessary, but I won't add more sections (I would write them on external documents).

1.4 License

All the document is FDL[5] (like the GPL from free software, but for documents). The source code is made with LyX (dn.en.lyx[6]), and uses the macros fitch.sty by Johan W. Klüwer. I have used the program latex2html (slightly patched) to create the web pages.

You have the right, among others, to modify it or to translate it to other languages which you know well, and also to redistribute it, sell it, and more.

2 Basic concepts

In logic one has to be perfectly clear of the meaning of each word. I will just remember what are and how to read the strange symbols used in this document.

[5] http://www.gnu.org/licenses/fdl.html
[6] http://www.danielclemente.com/logica/dn.en.lyx

2.1 Formalization

To formalize means writing an expression in a standard form which anyone can understand.

When working with logical algorithms, you can be thinking all the time in phrases like "*If I have a LCD screen but it has too many dead pixels, then I need another monitor*". You can, but they are too long. It's better to represent each action with a letter, and write the phrase using such letters along with simple words like *and*, *or*, *not*, or *then*.

For example, we have this vocabulary:

L: *have a LCD (Liquid Crystal Display) monitor*

P: *have all pixels working perfectly, with not too many fused ones*

M: *need a new monitor*

The phrase "*If I have a LCD screen but it has too many dead pixels, then I need another monitor*" is better expressed by "*if L and not P, then M*".

At natural deduction we will only use the version with letters, following these conditions:

- The letters (named *propositional letters*) are uppercase.
- Normally P, Q, R, S, ... are used, but anyone else is allowed.
- We use some special symbols for the operators *and*, *or*, *not* and *implication*.

2.2 Used symbols

To express the relation between one action and another, there exist some international icons. The basic operators you must know are $\vee$, $\wedge$, $\neg$, $\Rightarrow$. The others are more complex, but here I put all of them as a reference, to be able to find them if you were searching any of them.

Symbol	It's read...	Description
$\vee$	*or*	$A \vee B$ is true whenever one of the two, or both, are true.
$\wedge$	*and*	To make $A \wedge B$ true, both A and B have to be true.
$\neg$	*not*	$\neg A$ only is true when A is false.
$\Rightarrow$	*implies*	Shows consequence. The expression $A \Rightarrow B$ says that when A holds, so does B. In addition, $A \Rightarrow B$ is considered true except for the case A true and B false. To understand that, think of an A which implies B and ask yourself: *is it possible that A is true but not B?* Anyway, don't worry about that, it's not important right now.
$\Longleftrightarrow$	*if and only if*	$A \Longleftrightarrow B$ is the same as $(A \Rightarrow B) \wedge (B \Rightarrow A)$. It means that from A we can deduce B and viceversa, so they are equivalent.
$\square$	*false*	The empty square represents *false* (the binary *0*). Technically, it represents $\{\}$.
$\blacksquare$	*true*	The filled square represents *true* (the binary *1*). Technically, it represents $\{<>\}$.
$\exists$	*exists...*	$\exists x P x$ can be read *there exists an x such that P of x*. If in our domain, we can find an element (or more) which makes true the property P applied to that element, then the formula is true.
$\forall$	*for all...*	$\forall x P x$ can be read *for all x, P of x*. If all elements we are working with make the property P become true, then the formula is true.
$\vdash$	*then*	$\vdash$ is the symbol of the *sequent*, which is the way of saying "*when all this from the left happens, then it also happens all this from the right*". There exist valid sequents, like $P \wedge Q \vdash P$ or like $P \Rightarrow Q,\ Q \Rightarrow R,\ P \vdash P \wedge R$. But there are also invalid ones, like $P \Rightarrow Q,\ \neg P \vdash \neg Q$. The objective of natural deduction is to prove that a sequent is valid.
$\models$	*valid*	$\phi \models \varphi$ means that φ is logical consequence of ϕ, but when one writes $A \models B$, what we mean is that the sequent $A \vdash B$ is valid, that is, we could somehow prove it, and now is considered true for any interpretation of the predicate symbols.
$\not\models$	*invalid*	$\phi \not\models \varphi$ means that φ is not logical consequence of ϕ. If you can find a series of values (*model*) which make ϕ true but φ false, then invalidity is proven.
$\Vdash$	*satisfiable*	A set of formulas is satisfiable if there exists a series of values (*model*) which can make all of them true at the same time.
$\nVdash$	*unsatisfiable*	A set of formulas is unsatisfiable if there isn't any combination of variables (*model*) which can make all of them become true at the same time.

2.3 Precedence of operators

When you see an expression, you must be able to recognise what is it. For instance, $A \vee B \Rightarrow C$ is an implication (not a disjunction!), because $\Rightarrow$ is evaluated last (it has lower priority than $\vee$).

Here there are the operators, inversely sorted by priority:

- $\iff$
- $\Rightarrow$
- $\vee$ and $\wedge$ (they have the same priority)
- $\neg$

This means that $\neg$ is the one that most "*sticks*" to the symbol it has next. See this example about when and where are needed the parenthesis:

$P \vee \neg Q \Rightarrow R \wedge P \iff \neg(R \vee S) \wedge A \Rightarrow B$ is the same as $(\ (P \vee (\neg Q))\ \Rightarrow (R \wedge P)\ ) \iff (\ ((\neg(R \vee S)) \wedge A) \Rightarrow B\ )$

But don't panic, I won't use again expressions that long.

3 Natural deduction

Now I must explain what it is, how can it be done, and whether it has any practical use.

3.1 What it is for

Natural deduction is used to try to prove that some reasoning is correct ("*to check the validity of a sequent*", says theory). Example:

I tell you: "*In summer it's warm, and now we're in summer, so now it's warm*". You start doing calculations, and finally reply: "*OK, I can prove that the reasoning you just made is correct*". That is the use of natural deduction.

But it's not always so easy: "*if you fail a subject, you must repeat it. And if you don't study it, you fail it. Now suppose that you aren't repeating it. Then, or you study it, or you are failing it, or both of them*". This reasoning is valid and can be proven with natural deduction.

Remark that you don't have to believe nor understand what you are told. For example, I say that: "*Thyristors are tiny and funny; a pea is not tiny, so it isn't a thyristor*". Even if you don't know what am I talking about, or think that it is stupid (which it really is), you must be completely sure that the reasoning was correct.

So, given a supposition "*if all this happens, then all that also happens*", natural deduction allows us to say "*yes, that's right*". In logical language: if you are given a sequent $A \vdash B$, you can conclude at the end that it is $\models$ (*valid*). Then we write $A \models B$ (A has as consequence B).

3.2 What it is not for

It isn't suitable for proving *invalidity* of some supposition. I might say "*at daytime, it isn't night; and now it's daytime, so now it's also night*" and you may pass some time trying the rules of natural deduction, but obtaining nothing useful. After some time, you will intuitively discover that the reasoning might not be valid, and it's then when another methods -not natural deduction- should be tried in order to prove invalidity. They are explained later.

So, natural deduction only serves for proving validity, but not invalidity. What a pity, isn't it?

Neither does it serve to provide a good answer to the question "*What would happen if...?*". When we are to prove the validity of $A \vdash B$, we must think of things that would happen if A happened, and if we discover that one of these things is B, then we have finished. But we will never be able to give a complete and finite list of all those things.

3.3 Functioning

We are asked to prove the validity of $\Gamma \vdash S$, where Γ (that's *gamma*) is a group of formulas separated by commas, and S is a single formula.

We start assuming that all formulas in Γ are true, and, by continuous application of 9 concrete rules, we can go on discovering which other things are true. Our intention is to discover that S is true; so once we achieve that, we can stop working.

Sometimes we won't be able to extract truths from anywhere, and we will have to make suppositions: "*well, I'm not sure that $A \wedge B$ is always true, but if it holds that C, then it surely will be*". Then we have just discovered another truth: that $C \Rightarrow A \wedge B$.

As you can see, one has always to be thinking in where do we want to head to, because otherwise we could discover lots of things which are indeed true, but which we don't need at all. For instance, with $A \vee B$, $\neg A \vdash B$ we have to achieve the truth of B. We may discover that $\neg(A \wedge B)$, $A \vee B \vee C$, $(A \vee B) \Rightarrow \neg A$, and several other formulas, but what we really are interested in is B and nothing else. So, if you aren't following the right way towards the solution, you can make a mess.

3.4 Notation

There exist several ways to write the derivations done with natural deduction. I will use the Fitch style, because it's the one I used when learning, it's easy to understand, and occupies little space. It's something like this:

1	$P \Rightarrow Q$	
2	$Q \Rightarrow R$	
3	$\quad P$	H
4	$\quad Q$	E⇒ 1,3
5	$\quad R$	E⇒ 2,4
6	$\quad Q \wedge R$	I∧ 4,5
7	$P \Rightarrow Q \wedge R$	I⇒ 3,6

This is sufficient to prove the validity of $P \Rightarrow Q,\ Q \Rightarrow R \vdash P \Rightarrow Q \wedge R$.

That figure is to be done line by line, from top to the bottom. The numbers from the left show the number of each line, and are always in natural order.

The first lines contain each of the formulas which are written in the left part of the sequent. In this case, they are two: $P \Rightarrow Q$ and $Q \Rightarrow R$. From these we will have to achieve the formula $P \Rightarrow Q \wedge R$.

On each line we write what new thing we have just discovered to be true, and to the right we note how did we discover that. Those symbols from the right side (E and I) are the abbreviations of the names of the 9 rules. For example, here we can see *implication elimination* ($E \Rightarrow$), *conjunction introduction* ($I\wedge$), and *implication introduction* ($I \Rightarrow$). The numbers that go with them give us information about from where did we extract each necessary formula which is needed to apply the rule. They are line numbers, so, to be able to apply a rule, one has to use information only from the lines already written.

Finally, that vertical line which goes from line 3 to 6 it's a hypothesis (that's why we put H to the right). Everything which is inside it, is not always true, but only when happens P (the heading of the hypothesis, at line 3). So, all of the work we do inside the hypothesis cannot be used outside it, because it can't be assured to be always true.

The procedure finishes when we discover that it's true the formula at the right side of the sequent, in this case $P \Rightarrow Q \wedge R$ (it appears at the last line).

4 The derivation rules

Here they are defined and explained the nine basic rules which are used in natural deduction. Their objective is to tell us when and how can we add new formulas which continue being true.

Some examples (explained) are in the next section.

4.1 Iteration

This is a very simple rule:

$$
\begin{array}{ll|l}
\text{n} & A & \\
\hline
 & A & \text{IT n}
\end{array}
$$

Well, I know, written like this is a bit strange, but I put it this way to make it useful as the definition of the rule. What is contained in the above formulation is that if on line number n we have written A (whatever expression it is) then we have the option to write again A, but in the current line, and to justify that, we must write at the right $IT\ n$.

So, why would we want that? Well, for the moment, for nothing, but it will have its utility when we start working with hypothesis. Since a hypothesis is *closed*, all rules will have to work with formulas inside the hypothesis. If one of the formulas we want is just outside this hypothesis, we can copy it herein by using this rule called *iteration*.

Some people think that it's not necessary to waste a line this way, but it's a lot clearer when this rule is used. What isn't allowed is using it only to "*bring nearer*" some formula which is several lines far away: it isn't necessary to rewrite a line if we have it already written in the current derivation.

4.2 Conjunction introduction

The conjunction (that's the *and*) can be created easily:

$$
\begin{array}{lll}
\text{m} & A & \\
\text{n} & B & \\
\hline
 & A \land B & \text{I}\land \text{ m,n}
\end{array}
$$

You should be able to understand the meaning of figures like this one. When we put a long horizontal line, normally it's to separate the *premises* (top) from the *conclusion* (bottom). Premises are conditions which must be fulfilled in order to apply the rule, and conclusion (or *resolvent*) is the result of the application of the rule.

This rule says that if on one line we have written a truth, and on another line we have another one, also true, then we can write in just a line that both things are true. We must then note to the right the lines from where we picked the first and the second formulas.

This is pretty logic, isn't it? if we know that really *it's raining*, and that it's true that *now it's sunny*, then there's no problem in saying that *it's raining and sunny* (yes, at the same time). If something feels strange, it's not our fault; blame the one who told us that *it's raining* or *it's sunny*.

Remark that picking the lines reversed, you can obtain $B \land A$, and picking the same line you can achieve $A \land A$ and $B \land B$, which are also true.

4.3 Conjunction elimination

This is just the inverse operation of the previous one. It has two parts; firstly:

$$\begin{array}{lll} \text{n} & A \wedge B & \\ \hline & A & \text{E}\wedge \text{ n} \end{array}$$

And secondly, for the case you wanted B:

$$\begin{array}{lll} \text{n} & A \wedge B & \\ \hline & B & \text{E}\wedge \text{ n} \end{array}$$

So, you can separate in several lines the *conjunctands* of a conjunction (yes, I think it's used that strange word). That's why this rule is called *conjunction elimination*, because from one line which has conjunction symbols ($\wedge$) you can extract several which don't have it, supposedly trying to approach to the formula which we want proved.

4.4 Implication introduction

This is more interesting, since it allows doing something useful with hypothesis (those subdemonstrations which have a vertical bar to the left). It's:

$$\begin{array}{l|ll} \text{m} & A & \text{H} \\ \cline{2-2} \text{n} & B & \\ \hline & A \Rightarrow B & \text{I}\Rightarrow \text{ m,n} \end{array}$$

And what it does mean is that if we supposed something (call it A), and we just discovered (by using the rules) that supposing A made true B (whatever it is), then we have something clear: we can't assure that B always is true, but we can assure that A implies B, which is written $A \Rightarrow B$.

This allows us to end the subdemonstration and continue working with what we were doing before. Remember that you can't finish natural deduction inside a subdemonstration.

4.5 Implication elimination

This one is simpler than the previous, since it does not deal with suppositions but with facts:

$$\begin{array}{lll} \text{m} & A \Rightarrow B & \\ \text{n} & A & \\ \hline & B & \text{E}\Rightarrow \text{ m,n} \end{array}$$

Simply, if we are told that when A also happens B (that's what it means $A \Rightarrow B$), and they also tell us that now happens A, then we can assure that B.

This rule is also named *modus ponens*.

4.6 Disjunction introduction

The disjunction (that's the *or*) is very easy but not obvious:

$$
\begin{array}{lll}
\text{n} & A & \\
\hline
 & A \lor B & \text{I}\lor \text{ n}
\end{array}
$$

Well, to be exact, I will say that it's also available in the other order:

$$
\begin{array}{lll}
\text{n} & A & \\
\hline
 & B \lor A & \text{I}\lor \text{ n}
\end{array}
$$

That's wonderful, isn't it? If we know that "*it's Thursday*" we also know that "*it's Thursday or cows can fly*", "*it's Thursday or Friday*", or even "*it's Thursday... or not*". All of them are true.

But remember that, when talking, we tend to use *exclusive or* (*XOR*), which is true if one of the *disjunctands* is true but not when both of them are true at the same time. To a logician, the common phrase "*it's Thursday or Friday*" holds true under three situations: when *it's Thursday*, when *it's Friday*, and when *it's Thursday and Friday at the same time* (something difficult in the real world, but mathematicians are capable of doing all types of suppositions...).

4.7 Disjunction elimination

This is the most complicated rule, mainly because if we are given a phrase with *or*, like "*it's Thursday or Friday*", what can we deduce from it? That *it's Thursday*? No, it may be Friday. That *it's Friday*? No, it may be Thursday. That *it's Thursday or Friday*? Well, yes, but we already knew that...

The rule (now I explain it):

$$
\begin{array}{lll}
\text{m} & A \lor B & \\
 & \quad | \; A & \text{H} \\
\text{n} & \quad | \; C & \\
 & \quad | \; B & \text{H} \\
\text{p} & \quad | \; C & \\
\hline
 & C & \text{E}\lor \text{ m,n,p}
\end{array}
$$

We need more information besides the $A \lor B$. If, luckily, we happen to know $A \Rightarrow C$, and also $B \Rightarrow C$, then we do know what happens when $A \lor B$: both one option and the other drive us to C, so C is true.

This type of things only happen when the exercise is prepared so that the *disjunction elimination* appears, or when A and B are similar (then we will find some C which is implied by both).

Now an example: when I contracted my ADSL access to the Internet, it was with *Telefónica* or *Terra*, but I'm not sure of with which one (even they didn't know it). And in my country (Spain), any option was slow, awfully expensive, and loaded with problems. Typical Spanish. If we call all those *features* M (for mockery, misery, ...), then basically any Internet Service Provider was an M. Concretely, $Telefonica \Rightarrow M$ and $Terra \Rightarrow M$, so undoubtedly my ADSL had to be M, both if I had one or the other ISP. And indeed, I needed 9 months to fully subscribe to the service... Luckily all this happened now several years ago.

This derivation rule is also called *proof by cases*, since we have to check each possible case to see that they all involve the same conclusion.

4.8 Negation introduction

This one is nice and interesting:

m	A	H
n	B	
p	$\neg B$	
	$\neg A$	I$\neg$ m,n,p

If after supposing A, you achieved the conclusion that both B and $\neg B$ are true, you're not lost, since you just discovered another truth: that it's not possible for A to be true, that's it, $\neg A$ it's true.

For instance, I confess that *if I use Windows, I don't profit the time I am with my computer*. Since some years, *I do profit it*, so the conclusion is that *I don't use Windows*. To achieve that conclusion, the path that you would follow (maybe without thinking) is precisely the one that this rule needs: suppose that *I do use Windows*, in that case *I wouldn't profit my computer*. But I said that *I do profit it*, so that supposition must be wrong.

This procedure is called *reduction to the absurd* (*reductio ad absurdum*): suppose something to achieve a contradiction and be able to assert that what we supposed is false. It's specially useful if you start supposing *the contrary* of what you want to prove: if any contradiction can be discovered, then it's almost all done.

I should note that this is an *abuse of notation*: following all the laws of logic, it happens that each subdemonstration needs *one* conclusion (not two); and at the above hypothesis, it's not clear which one is the conclusion (B or $\neg B$?). The correct way to write it would be using *conjunction introduction* to say that $B \wedge \neg B$, and this one is the conclusion which shows the wrongness of the initial hypothesis. But my teachers didn't write that line.

4.9 Negation elimination

This one is too simple, but we also have to know it:

$$\frac{\text{n} \quad \neg\neg A}{A \qquad \text{E}\neg \text{ n}}$$

So, when we see the negation of the negation of something, we can take off these two following negations.

Remember that the negation of "*this is white*" is not "*this is black*" but "*this is not white*".

4.10 No more rules

That's it, there are no more basic rules. Well, there still exist some more to deal with *quantifiers* and two about *true* and *false*, which I will explain later, but with the former 9 we're able to try to prove the validity of any sequent in this document (except the ones with quantifiers...).

Remember again that there are no more rules: you can't change from $A \vee \neg A$ to $\blacksquare$ (*true*) directly, or from $\neg(A \vee B)$ to $\neg A \wedge \neg B$, nor use the distributive, associative or commutative property. You have to proceed always step by step; even the simple changes aren't allowed (currently). Why? Because probably they aren't that simple: you will understand it when having to prove things like that $A \vee \neg A$ is always true... (it's in the next section).

5 Explained exercises

Exercises from several levels, explained step by step. If you still want more examples (but without comments) look into the last section. What I'm trying to explain here is not the rules, but the way of thinking so that you can devise the magic idea which solves the problem.

This is what I more lacked when I had to study natural deduction.

5.1 A very simple one. $P,\ P \Rightarrow Q \vdash P \wedge Q$

The solution to $P,\ P \Rightarrow Q \vdash P \wedge Q$ is:

1	P	
2	$P \Rightarrow Q$	
3	Q	E$\Rightarrow$ 2,1
4	$P \wedge Q$	I$\wedge$ 1,3

Here we won't have to think much, we just have to use correctly the rules and their justifications.

Firstly, understand what has been told to us: they say that now happen *two* things, the first is that P and the second is that $P \Rightarrow Q$ (they are the two formulas written to the left of the $\vdash$). These two things we will note, one on each line, since at this demonstration they will always be true (liking it or not).

The goal of this demonstration is to know that $P \wedge Q$ is also true, as we have been told that when P and $P \Rightarrow Q$ are true, then $P \wedge Q$ also is, and we want to check if that's right. Finally we achieved it, since on the last line we see the $P \wedge Q$ written.

But how do we start? Remember where do we want to head to. If $P \wedge Q$ has to be true, then both P and Q should be true; let's attempt to prove that they really are.

P is true, since they said so, and we have it written on line 1.

But we weren't told that Q was true. What do we know about Q? Searching it on lines 1 and 2, the only we know about Q is that it's true when happens P (that's what says line 2). But P is true, so we can use one of the rules to deduce Q from the $P \Rightarrow Q$ and P. Remark what is the most important change when we go from $P \Rightarrow Q$ to Q: we stopped using the implication symbol; so the rule we will need is the one called *implication elimination*.

To use this rule, we look at its definition, and see that we have to write in a new line Q, and as a justification $E \Rightarrow\ 2, 1$ needs to be written. The E is from *elimination*, the $\Rightarrow$ means *implication*, the first number is the one from the line which does contain the implication ($P \Rightarrow Q$), and the second number is from the line which has the known truth (P). It's incorrect to write them reversed ($E \Rightarrow\ 1, 2$), since the definition of the rule says that the line which has the implication should be cited first.

We have just applied the rule, and now we know three truths: P, $P \Rightarrow Q$, and Q. They are all equally true. Now we're nearer to our objective, $P \wedge Q$, since we know that P and Q are true, so $P \wedge Q$ also has to be true (it's obvious). In the formula we search there's a conjunction sign ($\wedge$) which we don't have, so we need to use the *conjunction introduction* to be able to say that $P \wedge Q$ is true because P is and also Q. As a justification we write $I \wedge\ 1, 3$ (the line where it says P, and the one which says Q). Don't put $I \wedge\ 3, 1$, that would be to affirm that $Q \wedge P$, which is not what we're trying to prove.

Then we know 4 truths: P, $P \Rightarrow Q$, Q, and $P \wedge Q$. We could continue finding more things which are true, but we've already finished, since we had been told to prove that $P \wedge Q$ is true, and we just achieved that (in line 4). So that will be the last line, and we don't have to write anything else.

Ah, and an example of this derivation, but with words: "*now it's summer, and in summer it's warm. That's why now it's summer and it's warm*".

5.2 A bit more complicated. $P \wedge Q \Rightarrow R,\ Q \Rightarrow P,\ Q \vdash R$

Try yourself $P \wedge Q \Rightarrow R,\ Q \Rightarrow P,\ Q \vdash R$. Then look the solution:

1	$P \wedge Q \Rightarrow R$	
2	$Q \Rightarrow P$	
3	Q	
4	P	E$\Rightarrow$ 2,3
5	$P \wedge Q$	I$\wedge$ 4,3
6	R	E$\Rightarrow$ 1,5

The only way to achieve R is using the first formula, $P \wedge Q \Rightarrow R$, but we can only use it when $P \wedge Q$ is true, so we're going for that.

We know that $Q \Rightarrow P$ (line 2) and also Q (line 3), so we deduce P. Since P is now true and also Q, $P \wedge Q$ is too. Until now it's similar to the previous exercise.

Finally, we have $P \wedge Q \Rightarrow R$, and know that $P \wedge Q$, so we finish by saying R.

5.3 Starting to make suppositions. $P \Rightarrow Q,\ Q \Rightarrow R \vdash P \Rightarrow Q \wedge R$

This one, $P \Rightarrow Q,\ Q \Rightarrow R \vdash P \Rightarrow Q \wedge R$, is more interesting:

1	$P \Rightarrow Q$	
2	$Q \Rightarrow R$	
3	| P	H
4	| Q	E$\Rightarrow$ 1,3
5	| R	E$\Rightarrow$ 2,4
6	| $Q \wedge R$	I$\wedge$ 4,5
7	$P \Rightarrow Q \wedge R$	I$\Rightarrow$ 3,6

Note the following details:

- We aren't told any information about what does happen now (we don't have formulas like P, or $Q \wedge R$, etc.). They only say things like that *if happened P, then Q would also happen.*
- In the same way, what we must prove is not that *just now happens something*, but that *if it happened P, then Q and R would be true.*
- $P \Rightarrow Q \wedge R$ is an implication (*something implies something*), because operator $\Rightarrow$ has less priority than $\wedge$. It's a big error to understand that formula as $(P \Rightarrow Q) \wedge R$.

As the formula we want is an implication ($P \Rightarrow Q \wedge R$), we will have to use the *implication introduction*, but this rule needs having a subdemonstration (look it up at its definition).

It isn't hard to understand why: $P \Rightarrow Q \wedge R$ says that *if it happens* P*, then happens* $Q \wedge R$, so the first we should do is to suppose that P really does happen. Then we will have to discover that, in this case when P is true, it is also true $Q \wedge R$. When we get that, we will apply the rule and write everything politely: $P \Rightarrow Q \wedge R$.

For that reason, at line 3 we make an hypothesis (justified by the H at the right): suppose that P is true. Now we're starting a subdemonstration, where we will be able to use the truths that were on the father demonstration (lines 1 and 2 in this case), and also we can use P as if it were another truth.

We made this hypothesis aiming to know that $Q \wedge R$, so we deduce it similarly to the previous exercises. Notice that we use truths from inside and from outside the subdemonstration, and also that, while we haven't finished it, that vertical line to the left must be put.

In line 6 we now have $Q \wedge R$, which is what we were looking for. Using the *implication introduction* rule, we can go outside this subdemonstration by saying that *if the hypothesis is true, then what we deduced from it also is true*. We stop putting that vertical line, since $P \Rightarrow Q \wedge R$ is always true (it doesn't depend on whether P is true or not). The justification we used, $I \Rightarrow\ 3, 6$, says that 3 is the line where we made the supposition, and 6 the line where we discovered something interesting which happens when we make that supposition.

$P \Rightarrow Q \wedge R$ is what we wanted, so we have finished. We finish as always, since we're outside any subdemonstration.

5.4 Using iteration. $P \vdash Q \Rightarrow P$

This is a short one: $P \vdash Q \Rightarrow P$. Solution:

1	P	
2	$\quad$ | Q	H
3	$\quad$ | P	IT 1
4	$Q \Rightarrow P$	I$\Rightarrow$ 2,3

The way is clear: we have to suppose Q, and finally see that, in that case, P is true. The trick: P is always true, whether we suppose Q or not.

We must use *implication introduction*, but this needs a hypothesis, and, some lines below, the result of the supposition. Only then we can close the hypothesis.

So after opening it (line 2), we must do something to write down that P. Since we already have it written in line 1, we simply put P again and justify it with IT 1, which means "*I copied this from line 1*". The IT is for *iteration*.

We now fulfill the requirements to apply the rule, so we apply it, closing the subdemonstration, and we've ended.

5.5 Reduction to the absurd. $P \Rightarrow Q,\ \neg Q \vdash \neg P$

This is a very useful technique. Validity of $P \Rightarrow Q,\ \neg Q \vdash \neg P$ is proved with:

1	$P \Rightarrow Q$	
2	$\neg Q$	
3	$\mid\ P$	H
4	$\mid\ Q$	E$\Rightarrow$ 1,3
5	$\mid\ \neg Q$	IT 2
6	$\neg P$	I$\neg$ 3,4,5

What has to be achieved here is $\neg P$, which is the *negation of something*, so the rule which will help is the *negation introduction*, also known as *reduction to the absurd*.

The way to act will be to suppose the contrary of $\neg P$ (which is P) and find a contradiction (any). Supposing P will lead us to Q (by *implication elimination*), and, as we also have $\neg Q$, we can apply the rule. This $\neg Q$ should be inserted in the current subdemonstration with the *iteration* rule, so that it is together with the Q and *inside* the subdemonstration. Everything which is inside the subdemonstration is consequence of P, so it is important to see that both Q and $\neg Q$ also are.

For the *negation introduction*, the way to justify this rule is putting the line number of where does the (wrong) supposition start, and the numbers of the two lines where we saw the contradiction. The conclusion of this rule is the contrary of what we just supposed, in this case $\neg P$, so we can finish the derivation here.

This reasoning is actually made without much thinking. In words it would be something like: "*of course that$\neg P$, since if it were P then Q, and you say that $\neg Q$, so it can't be that P*".

5.6 With subdemonstrations. $P \Rightarrow (Q \Rightarrow R) \vdash Q \Rightarrow (P \Rightarrow R)$

Things get harder. Here's the solution to $P \Rightarrow (Q \Rightarrow R) \vdash Q \Rightarrow (P \Rightarrow R)$:

1	$P \Rightarrow (Q \Rightarrow R)$	
2	$\mid\ Q$	H
3	$\mid\ \mid\ P$	H
4	$\mid\ \mid\ Q \Rightarrow R$	E$\Rightarrow$ 1,3
5	$\mid\ \mid\ R$	E$\Rightarrow$ 4,2
6	$\mid\ P \Rightarrow R$	I$\Rightarrow$ 3,5
7	$Q \Rightarrow (P \Rightarrow R)$	I$\Rightarrow$ 2,6

But first: here we will only use the two rules that help adding and removing implications, since it's the only operator appearing in the formulas.

We want $Q \Rightarrow (P \Rightarrow R)$, so we will have to do a hypothesis Q inside of which we should prove that $P \Rightarrow R$. We now do that to simplify the problem: we open a subdemonstration at line 2. We won't close it until we discover that $P \Rightarrow R$ is true.

Now the problem is somehow easier. We just need to prove $P \Rightarrow R$, and we have two lines with two truths: the first says that $P \Rightarrow (Q \Rightarrow R)$, and the second says that Q.

How can we achieve the $P \Rightarrow R$? Well, as always: we must suppose P, and achieve that R is true, in some way. Even if it doesn't seem very simple, it's what must be done, since *implication introduction* works that way. So we're going to open another hypothesis, now supposing P, and let's see if we achieve R. This will be a hypothesis inside a hypothesis, but there's no problem in doing that.

After writing line 3, and being inside a *subsubdemonstration*, we have available that $P \Rightarrow (Q \Rightarrow R)$, that Q, and that P. We must prove R. Now it isn't that hard, is it? If we know that P, we can use *implication elimination* on line 1, and we will get the true formula $Q \Rightarrow R$. Since Q is also true (line 2), we can apply that rule again to discover that R.

We then see that supposing P leads us to the conclusion R, so we can write down $P \Rightarrow R$, which is what we wanted. Now we've gone outside the subsubdemonstration, and we're only under the supposition that Q is true. As we now see that this supposition implies the truth of the formula $P \Rightarrow R$, we can end this subdemonstration concluding that $Q \Rightarrow (P \Rightarrow R)$.

$Q \Rightarrow (P \Rightarrow R)$ is precisely what had to be proven, so we're finished.

5.7 One with proof by cases. $P \vee (Q \wedge R) \vdash P \vee Q$

It will be needed the most complex derivation rule: the *disjunction elimination.* $P \vee (Q \wedge R) \vdash P \vee Q$ solved:

1	$P \vee (Q \wedge R)$	
2	│ P	H
3	│ $P \vee Q$	I$\vee$ 2
4	│ $Q \wedge R$	H
5	│ Q	E$\wedge$ 4
6	│ $P \vee Q$	I$\vee$ 5
7	$P \vee Q$	E$\vee$ 1,3,6

You already know the rules, so I just explain the way of thinking of a human who doesn't know natural deduction but can think a little:

We need to know that $P \vee Q$ is always true. The expression from the left, $P \vee (Q \wedge R)$, can be made true because of two causes:

- if it's true because P is true, then $P \vee Q$ is true.
- if it's true because $Q \wedge R$ is true, then both Q and R are true, so $P \vee Q$ is true because of Q.

So, anyway, $P \vee Q$ is true.

Well, now we just need to translate all this to logical language, following the same order in which we thought that, and proceeding slowly.

We start proving one path, then the other, and finally we apply the *disjunction elimination*. To justify it we must write the line where the disjunction is in, and the two lines from inside each subdemonstration where we saw that both supposing one thing or supposing the other leads to the same result.

Notice that, even if we discovered that $P \Rightarrow P \vee Q$ and that $Q \wedge R \Rightarrow P \vee Q$, it isn't necessary to use *implication introduction* to keep this written down.

The hardest thing in *proof by cases* is to decide which will be the expression to prove in both cases. It must be exactly the same in both cases!

5.8 One to think. $L \wedge M \Rightarrow \neg P,\ I \Rightarrow P,\ M,\ I \vdash \neg L$

Try $L \wedge M \Rightarrow \neg P,\ I \Rightarrow P,\ M,\ I \vdash \neg L$ only thinking; then write it down on paper. It's something like:

1	$L \wedge M \Rightarrow \neg P$	
2	$I \Rightarrow P$	
3	M	
4	I	
5	│ L	H
6	│ $L \wedge M$	I∧ 5,3
7	│ $\neg P$	E⇒ 1,6
8	│ P	E⇒ 2,4
9	$\neg L$	I¬ 5,7,8

I will put it by words: "*if you use Linux and Mozilla as a browser, you avoid problems. In contrast, if you use Internet Explorer you will have problems. Now you use Mozilla, but also Internet Explorer sometimes. Consequently, I know that you don't use Linux*".

Maybe that seems evident: "*of course, since IE is not on Linux*", but notice that I never said that. There isn't the $I \Rightarrow \neg L$ anywhere.

The way in which you should think when you prepare this exercise is:

1. I need to prove$\neg L$, which is the negation of something. It can't be seen any rule of the form *something implies* $\neg L$ which allows me to obtain it directly. We should think of another way, for example *negation introduction* (*reduction to the absurd*): suppose that I do use Linux.

2. In the case when I use Linux, I would use both Linux and Mozilla, since I already used Mozilla before (it's the third truth which is written in the original problem).

3. Using both Linux and Mozilla, I wouldn't have computer problems, since $L \wedge M \Rightarrow \neg P$.

4. But I also used Internet Explorer (fourth truth), and since IE generates problems, I will have problems. P.

5. I got a contradiction: $\neg P$ and P. So, what's happening is that the supposition I did of using Linux is wrong: actually, $\neg L$.

Now you just have to follow the same procedure, but writing down each step, and using the derivation rules. What you will obtain is the figure above, which happens to have 5 procedure lines (the first 4 are only to copy the truths). Each line corresponds with the steps given here.

5.9 Left side empty. $\vdash P \Rightarrow P$

Proving $\vdash P \Rightarrow P$ is very easy and short:

1	P	H
2	P	IT 1
3	$P \Rightarrow P$	I$\Rightarrow$ 1,2

This case didn't occur before: now it seems that the left part of the sequent is empty. It means that we are not given any truth from which we can deduce $P \Rightarrow P$. Why? Because $P \Rightarrow P$ is always true, not depending on the value of P or other formulas.

It's more comfortable and interesting to solve one of these demonstrations, since you start working directly on the formula which you want to achieve. But beware, since there are some *absolute truths* (always true) very hard and long to prove.

Note down this: whenever the left side is empty, you must start doing a hypothesis (what else could you do?).

To achieve $P \Rightarrow P$ we do as always: suppose that P and try to see that P is true. Since we just supposed it on the first line, we can use the *iteration* rule to copy it inside, and we finish the subdemonstration by using *implication introduction*. And we're done, in only three lines.

Remark that $P \Rightarrow P$ is true because $\blacksquare \Rightarrow \blacksquare$ and $\square \Rightarrow \square$. Well, and furthermore, remember also that $\square \Rightarrow \blacksquare$, but $\blacksquare \not\Rightarrow \square$.

5.10 Suppose the contrary. $\vdash \neg(P \wedge \neg P)$

Another simple one, $\vdash \neg(P \wedge \neg P)$. It's done this way:

1	$\mid P \wedge \neg P$	H
2	$\mid P$	E$\wedge$ 1
3	$\mid \neg P$	E$\wedge$ 1
4	$\neg(P \wedge \neg P)$	I$\neg$ 1,2

We all know that two contrary things can't happen at the same time, but, how can this be proved? We must use the *reduction to the absurd*:

Suppose that it does happen $P \wedge \neg P$. Then happen both P and $\neg P$, both at the same time, which is a contradiction. So, the supposition we just done can't be true: it's false. This way we can prove $\neg(P \wedge \neg P)$.

When you see something so *clear* and *obvious* as $\neg(P \wedge \neg P)$, then its contrary will be *clearly* false and absurd. So, it won't be too difficult to see that it doesn't hold and that it contradicts itself. Once done, we can assure that the original formula is true since its contrary is false.

5.11 This one seems easy. $\vdash P \vee \neg P$

Let's see if $\vdash P \vee \neg P$ is as easy as some say:

1	$\mid \neg(P \vee \neg P)$	H
2	$\mid \mid P$	H
3	$\mid \mid P \vee \neg P$	I$\vee$ 2
4	$\mid \mid \neg(P \vee \neg P)$	IT 1
5	$\mid \neg P$	I$\neg$ 2,3,4
6	$\mid \mid \neg P$	H
7	$\mid \mid P \vee \neg P$	I$\vee$ 6
8	$\mid \mid \neg(P \vee \neg P)$	IT 1
9	$\mid \neg\neg P$	I$\neg$ 6,7,8
10	$\mid P$	E$\neg$ 9
11	$\neg\neg(P \vee \neg P)$	I$\neg$ 1,5,10
12	$P \vee \neg P$	E$\neg$ 11

One of the simplest but longest I found. It seems even unnecessary to prove this, since everyone knows that between "*today it's Thursday*" and "*today it's not Thursday*", one of them is true (they can't be both be false at the same time).

We could start by thinking in the *proof by cases* method, since from P we can extract $P \vee \neg P$, and from $\neg P$ we can extract $P \vee \neg P$, so, the same formula. But this doesn't help, since *proof by cases* is the *disjunction elimination*, and we don't have any disjunction to eliminate; in fact, we also don't have the truth formula $A \vee B$ in which $A \Rightarrow C$ and $B \Rightarrow C$, as the rule needs. Actually, we don't have any formula which we know it's true (as the left part of the sequent is empty).

We know that we must start with a hypothesis (there's no alternative). Since it's rather clear that $P \vee \neg P$ is true, it may also be easy to prove that its contrary, $\neg(P \vee \neg P)$, is false. So we will use *reduction to the absurd*: doing that supposition on line 1, we must achieve a contradiction, any one.

I proposed myself to achieve the contradiction$\neg P$ and P. But we don't have any of these formulas; how can we obtain them? Doing *reduction to the absurd* again is an option: to see that $\neg P$, suppose that P and get a contradiction. As we did in another occasions, it's very useful to profit the capabilities of the *disjunction introduction*: having supposed P, we can convert it to $P \vee \neg P$ to search our contradiction. As we have the $\neg(P \vee \neg P)$ at the top, we can use it to finish by demonstrating $\neg P$. We can do the same to prove P, but this time supposing $\neg P$.

Having obtained P and $\neg P$ after supposing $\neg(P \vee \neg P)$, we see that this formula can't be true, so its negation, $\neg\neg(P \vee \neg P)$, is. By *negation elimination*, we get our searched formula: $P \vee \neg P$.

I did it this way to make it rather symmetrical, but it can be shorter if we search another contradiction, for instance $P \vee \neg P$ and $\neg(P \vee \neg P)$. Then it would be like this:

1	$\mid \neg(P \vee \neg P)$	H
2	$\mid \mid P$	H
3	$\mid \mid P \vee \neg P$	I$\vee$ 2
4	$\mid \mid \neg(P \vee \neg P)$	IT 1
5	$\mid \neg P$	I$\neg$ 2,3,4
6	$\mid P \vee \neg P$	I$\vee$ 5
7	$\mid \neg(P \vee \neg P)$	IT 1
8	$\neg\neg(P \vee \neg P)$	I$\neg$ 1,6,7
9	$P \vee \neg P$	E$\neg$ 8

5.12 An interesting one. $P \vee Q,\ \neg P \vdash Q$

Another which seems easy: $P \vee Q,\ \neg P \vdash Q$. Let's see:

1	$P \vee Q$	
2	$\neg P$	
3	$\mid P$	H
4	$\mid \mid \neg Q$	H
5	$\mid \mid \neg P$	IT 2
6	$\mid \mid P$	IT 3
7	$\mid \neg\neg Q$	I$\neg$ 4,5,6
8	$\mid Q$	E$\neg$ 7
9	$\mid Q$	H
10	$\mid Q$	IT 9
11	Q	E$\vee$ 1,8,10

It's very easy to understand by anyone: it holds $P \vee Q$, but P is false, so the truth is Q.

It can be done in several ways, but at some time you will have to use *disjunction elimination* to do something with the $P \vee Q$. We're going to prove that both P and Q lead to the same place, which will be our target formula Q (since it's possible, let's go directly for Q).

We open subdemonstration supposing that P, and we must see that Q. It isn't too hard since we have $\neg P$ on line 2; this helps contradicting anything we want. Since what we're searching is Q, we suppose $\neg Q$ and by *reduction to the absurd* we obtain $\neg\neg Q$, which is Q.

The other path, when we suppose Q true, leads us directly to Q.

In conclusion, both paths go to Q and by *disjunction elimination* we get the proof that Q is always certain.

5.13 I had this one in an exam. $A \vee B,\ A \Rightarrow C,\ \neg D \Rightarrow \neg B \vdash C \vee D$

In the final exam of *ILO* they were asking $A \vee B,\ A \Rightarrow C,\ \neg D \Rightarrow \neg B \vdash C \vee D$, and I needed a very very long time until I got it:

1	$A \lor B$	
2	$A \Rightarrow C$	
3	$\neg D \Rightarrow \neg B$	
4	$\mid A$	H
5	$\mid C$	E$\Rightarrow$ 2,4
6	$\mid C \lor D$	I$\lor$ 5
7	$\mid B$	H
8	$\mid \mid \neg D$	H
9	$\mid \mid \neg B$	E$\Rightarrow$ 3,8
10	$\mid \mid B$	IT 7
11	$\mid \neg\neg D$	I$\neg$ 8,9,10
12	$\mid D$	E$\neg$ 11
13	$\mid C \lor D$	I$\lor$ 12
14	$C \lor D$	E$\lor$ 1,6,13

Remark that the result we're searching, $C \lor D$, is a disjunction. Since you already know the *disjunction introduction*, you could simply search C, and then use that rule to get $C \lor D$. Or if with C didn't work, you could try with D, since if D is true, then $C \lor D$ also is, and we're done.

Unfortunately, C is not always true, and D also isn't always true (on the other hand, $C \lor D$ is always true, and that's what we're trying to prove). After seeing this, we must search another method which works with the two formulas, C and D, at the same time, since it seems that if we take only one without looking at the other, then it does not provide much information.

To use the $A \lor B$ we must use *proof by cases*. We will try to see that both A and B lead to $C \lor D$, since if we can do that, we will have finished.

A implies C, and if C is true then $C \lor D$ also is, so A implies $C \lor D$.

With B, what we know doesn't relate it to C but to D. We want $C \lor D$. Hardly we will make true $C \lor D$ because of C, so we will try to make true just the D. To do so, we will use *reduction to the absurd*: suppose that D is false, then it holds that $\neg B$ thanks to the formula on line 3. But we were under the supposition that B was true, so our hypothesis $\neg D$ can't be true, thus D is true, and so is $C \lor D$.

Since $A \lor B$ is true, and both paths lead to $C \lor D$, we finally see that $C \lor D$ is always true.

If you are skilled working with logical formulas, you will have seen that $\neg D \Rightarrow \neg B$ is $B \Rightarrow D$. This simplifies the problem and helps understanding it faster. But anyway, you can't change $\neg D \Rightarrow \neg B$ to $B \Rightarrow D$ directly, you would have to do it step by step.

5.14 A "short" one. $A \iff B \vdash (A \wedge B) \vee (\neg A \wedge \neg B)$

Seems easy: if two expressions are equivalent, it's because they are both true, or both false. I could prove the validity of $A \iff B \vdash (A \wedge B) \vee (\neg A \wedge \neg B)$ this way:

1	$(A \Rightarrow B) \wedge (B \Rightarrow A)$	
2	$\mid\ \neg(A \vee \neg A)$	H
3	$\mid\ \mid\ A$	H
4	$\mid\ \mid\ A \vee \neg A$	I$\vee$ 3
5	$\mid\ \mid\ \neg(A \vee \neg A)$	IT 2
6	$\mid\ \neg A$	I$\neg$ 3,4,5
7	$\mid\ A \vee \neg A$	I$\vee$ 6
8	$\mid\ \neg(A \vee \neg A)$	IT 2
9	$\neg\neg(A \vee \neg A)$	I$\neg$ 2,7,8
10	$A \vee \neg A$	E$\neg$ 9
11	$\mid\ A$	H
12	$\mid\ A \Rightarrow B$	E$\wedge$ 1
13	$\mid\ B$	E$\Rightarrow$ 12,11
14	$\mid\ A \wedge B$	I$\wedge$ 11,13
15	$\mid\ (A \wedge B) \vee (\neg A \wedge \neg B)$	I$\vee$ 14
16	$\mid\ \neg A$	H
17	$\mid\ \mid\ B$	H
18	$\mid\ \mid\ B \Rightarrow A$	E$\wedge$ 1
19	$\mid\ \mid\ A$	E$\Rightarrow$ 18,17
20	$\mid\ \mid\ \neg A$	IT 16
21	$\mid\ \neg B$	I$\neg$ 17,19,20
22	$\mid\ \neg A \wedge \neg B$	I$\wedge$ 16,21
23	$\mid\ (A \wedge B) \vee (\neg A \wedge \neg B)$	I$\vee$ 22
24	$(A \wedge B) \vee (\neg A \wedge \neg B)$	E$\vee$ 10,15,23

Firstly: we can't write $A \iff B$ since we don't have rules for $\iff$. Since it is seldom used, when a $\iff$ appears we are allowed to change it to $(A \Rightarrow B) \wedge (B \Rightarrow A)$, which is the same.

Well, this is the only idea I had... I leave as an exercise to find a shorter way to do it (if it does exist). What I did here was to write down that $A \vee \neg A$ is true (we already did this exercise, and here I just copied the same steps). Once I know that $A \vee \neg A$ holds, I see that both the case A and the case $\neg A$ lead to the same formula, which is the solution.

6 Wrong things

Common errors you mustn't do. Remember that a logic professor will correct your exercises with a *true* or a *false*, so learn to do this perfectly.

6.1 Introduction and elimination of ***"what it would be nice to have"***

The rules like *introduction* and *elimination* are not to allow you writing anything you want, but to help you using or creating a formula with a concrete operator.

That's why, if you have P, you can't say "*now I do* negation introduction *and get* $\neg P$*, which is what I needed*". There are some requisites for each rule, and if you don't fulfill them, you can't apply that rule.

For instance: the rule *implication elimination* doesn't allow to use the formulas in the first line this way:

1	$P \Rightarrow Q \wedge R$	
2	$Q \wedge R$	E$\Rightarrow$ 1,1

$\otimes$ *INCORRECTO* $\otimes$

To be able to do this, we would need to be sure that P is always true; then we could apply the rule, correctly writing the line numbers.

6.2 Iterate something from a non attainable subdemonstration

Inside the main demonstration (which goes from the first line to the last), we can open *child demonstrations* (*subdemonstrations*). Inside any subdemonstration we can also have a *subsubdemonstration*, which would have as father the subdemonstration and as grandfather the main demonstration.

To understand this, here is the solved example $A \vee B,\ A \Rightarrow C,\ \neg D \Rightarrow \neg B \vdash C \vee D$:

1	$A \vee B$	
2	$A \Rightarrow C$	
3	$\neg D \Rightarrow \neg B$	
4	$\mid\ A$	H
5	$\mid\ C$	E$\Rightarrow$ 2,4
6	$\mid\ C \vee D$	I$\vee$ 5
7	$\mid\ B$	H
8	$\mid\ \mid\ \neg D$	H
9	$\mid\ \mid\ \neg B$	E$\Rightarrow$ 3,8
10	$\mid\ \mid\ B$	IT 7
11	$\mid\ \neg\neg D$	I$\neg$ 8,9,10
12	$\mid\ D$	E$\neg$ 11
13	$\mid\ C \vee D$	I$\vee$ 12
14	$C \vee D$	E$\vee$ 1,6,13

Well, any demonstration can only access the formulas from inside itself, inside its father, inside the father of its father, inside the father of the father of its father, ... All these are called *ancestors*, so: *a demonstration can access itself and its ancestors.*

For this reason, it we are on line 10, the derivation rules can use formulas from the following places:

- the current demonstration (lines 8 and 9 currently).
- father demonstration of the 8-10 one, so, from line 7.
- from the demonstration father of the one which starts at line 7, that's it, lines 1 to 3.

Bet never we could use the formulas from lines 4 to 6, which is the demonstration *uncle* of the current one (brother of its father), because all that demonstration is based on the hypothesis that A (line 4), and we're not doing that supposition anymore.

In logical language, one says that a formula A is *actual* at formula B if being in B we can use A. For this to be true, A must have been written before B, and some ancestor of B must be father of A.

So, to prove $P \wedge Q$ we can't do this:

1	$\mid P$	H	
2	$\mid \mid Q$	H	$\otimes$ *INCORRECTO* $\otimes$
3	$\mid \mid P \wedge Q$	I$\wedge$ 1,2	
4	$P \wedge Q$	IT 3	

6.3 Misplace parenthesis

When I wrote the definitions of the rules, I used the letters A and B, but these can represent any expression.

For instance, here we do *negation introduction*, in which -following the rule- we suppose some formula A, attain a contradiction, and we conclude $\neg A$, so, the original formula, but negated. Let's see:

1	$\mid P \Rightarrow Q$	H	
$\dots$	$\mid \dots$		$\otimes$ *INCORRECTO* $\otimes$
7	$\neg P \Rightarrow Q$	I$\neg$ 1,$\dots$	

I think it's clear that the A which appears in the rule represents to $P \Rightarrow Q$ in this example. The problem comes when we do the $\neg A$. The negation of $P \Rightarrow Q$ is not $\neg P \Rightarrow Q$, but $\neg(P \Rightarrow Q)$. It's necessary that parenthesis because if not present, the negation affects only P.

If you don't know when to put parenthesis, always put them, and then try to remove the unneeded ones. For instance, if you must write that $\neg P \vee R$ implies $R \wedge Q$, put parenthesis around each expression and thus write $(\neg P \vee R) \Rightarrow (R \wedge Q)$. This way, there are absolutely no errors. Now learn when is it possible to remove parenthesis, and take away all that you can. In this case, both can be suppressed and it remains $\neg P \vee R \Rightarrow R \wedge Q$.

6.4 Finish inside a subdemonstration

You can't finish the deduction inside a subdemonstration. The last line can't have any vertical line to the left.

The reason is that everything from inside the subdemonstration is valid only when the supposition is really true, and what the original problem asks is to prove that the formula at the right of the $\vdash$ is *always* true.

Here's a sample of what can be tried by someone very astute who wants to prove $P \wedge Q$:

1	$\mid P$	H	
2	$\mid \mid Q$	H	$\otimes$ *INCORRECTO* $\otimes$
3	$\mid \mid P \wedge Q$	I$\wedge$ 1,2	

We supposed P, and also Q. In that case, of course it's true that $P \wedge Q$, but only in that case. We can't affirm to anyone that $P \wedge Q$ is always true. So, we should start closing the two demonstrations (first the inner one, and then the outer one) to extract some conclusion which is always valid.

Neither could we do that *iteration* thing at line 4. I already explained this before.

6.5 Skip steps

Even if you know equivalences between formulas, it's much better if you don't use them. For instance, if you have to write the negation of $\neg P$, don't write P directly, but $\neg\neg P$.

Understand that not everything is so obvious as it seems, and that someone may ask you to prove things like $P \vdash \neg\neg P$, where if you could use those simplifications, you would do almost no work.

Another example: going from $\neg(A \vee B)$ in one line to having $\neg A \wedge \neg B$ in the next can't be justified with any of the 9 rules. But if you succeed in proving and understanding that $\neg(A \vee B) \vdash \neg A \wedge \neg B$, then maybe you can add that as an additional rule to use in future demonstrations. I will give some of these in the next section.

7 Making it harder

Here I will finish the explanation of everything else that I was taught about natural deduction (even if we didn't use it very much). The quantifiers thing is really important, but more complex.

7.1 Rules about truth and false

We can work directly with the values $\blacksquare$ (*true*) and $\square$ (*false*), and also introduce or eliminate them from our demonstration following some easy rules.

7.1.1 Truth introduction

This is the easiest one:

$$\frac{}{\blacksquare} \quad \mathrm{I}\blacksquare$$

So, always, and with no requirements, we can write down that $\blacksquare$ is true, because it really is.

7.1.2 False elimination

A funny one:

$$\frac{n \quad \square}{A \qquad E\square\ n}$$

Explanation: if we achieved the conclusion that $\square$ is true, then we have already achieved a state where we can invent anything and affirm that it's true; at least, as true as the idea of $\square$ (*false*) being true.

This rule is called *ex falso quodlibet sequitur*, something like "*from false can follow anything*".

7.2 Rules about quantifiers

We're too limited if we can only say things like P, Q, R, ... to translate phrases to logical language. Quantifiers will allow us to do much more.

7.2.1 What's that

I won't be able to explain everything, since various previous concepts are needed, but I will try to say a little about them. First, some changes:

Now we won't talk just about general facts (*it rains*, *it's warm*, etc.), but we will have a *domain* of known things, and we will have to say which property is true for each element.

For instance: we have the domain $\{p,\ t,\ r\}$, which represent respectively to *PROLOG* (a logical programming language), a *telephone*, and a *radio*. p, t, r.

We also add a *predicate letter* (they're not called *propositional letters* anymore) E, which will have the following meaning: when we write Ex (read "*E of x*", but written together) we mean that *x is an electronic device*. We will also have Sx to say that *x is a piece of software*, and Tx which will mean that *x is a text processor*.

Now we know that are true Et, Er, Sp and nothing else.

Quantifiers make possible to write truths referring to some elements from the domain. There exist two quantifiers:

- Universal quantifier: $\forall$. When we put $\forall xPx$ ("*for all x, P of x*"), we mean that all elements on the domain make true the property P.
- Existential quantifier: $\exists$. $\exists xPx$ ("*there exists x such that P of x*") we mean that at least one element from the domain makes true the property P.

For instance, now are true the following formulas: $\forall x(Ex \vee Sx)$, $\neg\exists xTx$, $\forall x(Tx \Rightarrow \neg Ex)$, $\exists xEx \wedge \exists xSx$ and several more. Quantifiers have the same priority as the operator $\neg$.

The rules explained here will work only with *free substitutions*. Sorry for not saying what that means, but I don't want to go out of topic.

7.2.2 Existential introduction

If we see a proof of its existence, we can say that a property is true for some element:

$$
\begin{array}{lll}
\text{n} & A\{t/x\} & \\
\hline
 & \exists x A & \text{I}\exists \text{ n,t}
\end{array}
$$

That $A\{t/x\}$ is a *substitution* (maybe read "*t over x*" and is done by changing x to t).

This rule says that if we see At, where t is any element, we can say that $\exists x Ax$, because we know that when x is t then the formula is true.

7.2.3 Existential elimination

Extracting some truth from a $\exists x Px$ is tricky, but it's done this way:

$$
\begin{array}{llll}
\text{m} & \exists x A & & \\
\text{n} & \quad\mid\ A\{a/x\} & & \text{H} \\
\text{p} & \quad\mid\ B & & \\
\hline
 & B & & \text{E}\exists \text{ m,n,p,a}
\end{array}
$$

So, if one of the A implies B, then we know that B, since we know that one of the A is true. No a should appear in B nor in any attainable hypothesis (sorry for the cryptical phrases, they are part of the theory).

7.2.4 Universal introduction

Well, this one is easy:

$$
\begin{array}{lll}
\text{n} & A & \\
\hline
 & \forall x A & \text{I}\forall \text{ n}
\end{array}
$$

So, if we know that A is always true, then it is true for any value of x. No *free* x should appear in any attainable hypothesis.

7.2.5 Universal elimination

Another easy one:

$$
\begin{array}{lll}
\text{n} & \forall x A & \\
\hline
 & A\{t/x\} & \text{E}\forall \text{ n,t}
\end{array}
$$

If we know that A always holds for any element, then we can select an element (anyone) and we also know that A is true on that element.

7.2.6 Examples

At the last section there are several examples with quantifiers, but without explanations. Probably you will have to look for them in some logic book if you're trying to understand them.

7.3 Derived rules

In some books or tutorials more rules are allowed (apart from the basic 9) in order to deal with formulas more easily. They represent an abstraction: stop working in the details to dedicate our work in more complex problems (it's like the *high level* programming languages).

If you decide to use them, you will lose a lot of interesting work to do, but you will finish faster. My advice is to only use a rule if you know how to prove its validity by using the 9 basic rules.

Some of the ones I found at several places are:

- *Law of double negation*: allows changing A to $\neg\neg A$ and viceversa.
- *Modus Tollens*: having $A \Rightarrow B$ and $\neg B$, then $\neg A$.
- *Disjunctive syllogism*: if $A \vee B$ and $\neg A$, then B. And if $A \vee B$ and $\neg B$, then it's A.
- *Elimination of* $\neg\Rightarrow$: if you have $\neg(A \Rightarrow B)$, then happen both A and $\neg B$.
- *Elimination of* $\neg\vee$: if you have $\neg(A \vee B)$, then $\neg A$, and also $\neg B$.
- *Elimination of* $\neg\wedge$: if you have $\neg(A \wedge B)$, then $\neg A \vee \neg B$.
- *Theorems which you can use when you want*: $A \Rightarrow A$, $A \vee \neg A$, $\neg(A \wedge \neg A)$ and more.
- *Change of equivalent formulas*: if $A \iff B$, then where it says A you can put B and viceversa.

There are lots more; but if someone requests you an exercise, they will tell you which rules are allowed and which not (for instance, in class we were allowed to use only the basic ones).

8 Extra

If you already knew everything I explained, or have doubts about other topics unrelated to the way of doing natural deduction, stay at this section.

8.1 Why is it called natural deduction?

Because the procedures to be applied are the same that the ones people use when they think.

You can see that at most solved exercises in this manual. Express the sequents by words, tell them to someone, and after some time he/she will be saying "*of course it's like that, since ...*". You will see that anyone is able to explain how to use some of the 9 derivation rules, even without knowing their name or existence.

For this reason, to discover the solution to a natural deduction problem, forget about *introduction* and *elimination* rules, and think normally, changing the letters to simple actions if necessary. It can help to think of concepts like *it rains*, *it doesn't rain*, *it's sunny*, *I don't get wet*, ... since they are short words and, moreover, everyone has a clear understanding of what does happen when it rains, and can rapidly relate *not getting wet* with *being sunny and not raining*, or even more complex formulas.

8.2 Is the solution unique?

No. The more complex the exercise, the more ways to solve it correctly there are. In the section about explained exercises, I already gave several solutions to one of them.

Of course, you can start deducing things which are absolutely unneeded, and you will achieve a solution different from the others. But it's better to try to solve each exercise in the minimum number of steps.

8.3 Other ways to prove validity

Natural deduction is a way to prove the validity of a sequent, but there exist still more. Some of them are:

8.3.1 Brute force

We can list all the possible combinations of values for each variable, and check that, for each combination, if the left part of the sequent is true then the right part is also true.

Working with n variables, you will have to test 2^n cases.

The problem here are quantifiers, since now there's a domain involved. And we're not able to list some of the possible existing domains, since a domain can have infinite elements.

8.3.2 Refutation theorem

Refutation theorem says that $\Gamma \vDash A \Longleftrightarrow \nvDash \Gamma, \neg A$.

In words: the set of formulas Γ (*gamma*) has as consequence A *if and only if* the system composed by Γ together with $\neg A$ is unsatisfiable.

That about proving *unsatisfiability* is a different topic, and a rather long one, like its name suggests. One of the easiests methods to do that is using clause *resolution* trees.

8.4 How to prove invalidity

Natural deduction provides a method to demonstrate that a reasoning is correct, but, how can you prove that a reasoning is non-correct? It can't be done with natural deduction.

We are in this situation: we have sequent $\Gamma \vdash A$, and we think that there exists a *model* (set of values) which make true Γ -*gamma*- but not A. Well, then we just have to find it to prove that the sequent is invalid. This model is called *countermodel*, and we can find it in several ways. I think that the simplest one is *intuitively*: start trying different values which we regard as possible countermodel, until we find a good one.

For instance, $\neg P \Rightarrow \neg Q, \ \neg Q \vdash \neg P \vee Q$ is invalid ($\not\models$), since when P is true and Q is false, the left part (*antecedent*) becomes true but the right part (*consequent*) is false, so $\neg P \vee Q$ is not a consequence of that from the left part.

8.5 Create your own exercises

If you have already read and learnt all the examples from this document, you did wrong! Now you lack exercises to solve by yourself.

You can invent sequents and try to prove that they are valid; the problem then is that if they are not, you will waste your time trying to prove their validity in vain. So you must think only of valid sequents, and then prove them correctly.

Some methods I know to do that are:

- If A and B are the same formula, but written in some different ways, then try proving $A \models B$ or $B \models A$.
- Take a truth and prove it. For instance: $\vdash P \wedge P \Rightarrow P \vee P$.
- Take a lie, negate it, and try to prove that formula. Example: $\neg(A \wedge (A \Rightarrow B) \wedge \neg B)$. This method will make you practise *reduction to the absurd*.
- Convert some formula to its *conjunctive normal form* (so it is expressed like *something* $\wedge$ *something* $\wedge$ ... $\wedge$ *something*). Then you have several formulas which are all true at the same time: each of the conjunctands. You can select one of them and assert that when the original formula is true, then that conjunctand also is.
- Take several formulas at random, and suppose that all of them are true simultaneously. To do that, write their conjunction (*one* $\wedge$ *other* $\wedge$ *other* $\wedge$ ...). This big formula can be modified with the above methods to find some of its consequences. All this will be useful to practise natural deduction with *several* true formulas at the left part of the sequent.

8.6 Programs which do natural deduction

Is there any computer program which can do all these things I explained, but without having to think or work at all? Well, I really don't know; I didn't find any. All the examples in here were done manually.

Maybe you can make tools like seqprover[7] or pandora[8] work. I didn't succeed, and the few programs I found were uncomplete or were just projects. Probably that type of program would be hard to do, since deduction is *natural* (more appropriate for human brains). Anyway, computers might apply brute force...

What you can try, and works, is a game[9] similar to domino, with which you can prove sequents by using coloured tiles. It requires some learning.

9 Examples, lots of examples

And finally, here is a collection of several examples (without explanation). It was me who solved them, so if you find errors, tell me about it.

The first 14 were explained (by words) on section 5.

9.1 $P,\ P \Rightarrow Q \vdash P \wedge Q$

1	P	
2	$P \Rightarrow Q$	
3	Q	E⇒ 2,1
4	$P \wedge Q$	I∧ 1,3

9.2 $P \wedge Q \Rightarrow R,\ Q \Rightarrow P,\ Q \vdash R$

1	$P \wedge Q \Rightarrow R$	
2	$Q \Rightarrow P$	
3	Q	
4	P	E⇒ 2,3
5	$P \wedge Q$	I∧ 4,3
6	R	E⇒ 1,5

[7] http://bach.istc.kobe-u.ac.jp/seqprover/
[8] http://www.doc.ic.ac.uk/ yg/projects/AI/prover.html
[9] http://www.winterdrache.de/freeware/domino/

9.3 $P \Rightarrow Q,\ Q \Rightarrow R \vdash P \Rightarrow Q \wedge R$

1	$P \Rightarrow Q$	
2	$Q \Rightarrow R$	
3	| P	H
4	| Q	E$\Rightarrow$ 1,3
5	| R	E$\Rightarrow$ 2,4
6	| $Q \wedge R$	I$\wedge$ 4,5
7	$P \Rightarrow Q \wedge R$	I$\Rightarrow$ 3,6

9.4 $P \vdash Q \Rightarrow P$

1	P	
2	| Q	H
3	| P	IT 1
4	$Q \Rightarrow P$	I$\Rightarrow$ 2,3

9.5 $P \Rightarrow Q,\ \neg Q \vdash \neg P$

1	$P \Rightarrow Q$	
2	$\neg Q$	
3	| P	H
4	| Q	E$\Rightarrow$ 1,3
5	| $\neg Q$	IT 2
6	$\neg P$	I$\neg$ 3,4,5

9.6 $P \Rightarrow (Q \Rightarrow R) \vdash Q \Rightarrow (P \Rightarrow R)$

1	$P \Rightarrow (Q \Rightarrow R)$	
2	| Q	H
3	| | P	H
4	| | $Q \Rightarrow R$	E$\Rightarrow$ 1,3
5	| | R	E$\Rightarrow$ 4,2
6	| $P \Rightarrow R$	I$\Rightarrow$ 3,5
7	$Q \Rightarrow (P \Rightarrow R)$	I$\Rightarrow$ 2,6

9.7 $P \vee (Q \wedge R) \vdash P \vee Q$

1	$P \vee (Q \wedge R)$	
2	$\mid\ P$	H
3	$\mid\ P \vee Q$	I$\vee$ 2
4	$\mid\ Q \wedge R$	H
5	$\mid\ Q$	E$\wedge$ 4
6	$\mid\ P \vee Q$	I$\vee$ 5
7	$P \vee Q$	E$\vee$ 1,3,6

9.8 $L \wedge M \Rightarrow \neg P,\ I \Rightarrow P,\ M,\ I \vdash \neg L$

1	$L \wedge M \Rightarrow \neg P$	
2	$I \Rightarrow P$	
3	M	
4	I	
5	$\mid\ L$	H
6	$\mid\ L \wedge M$	I$\wedge$ 5,3
7	$\mid\ \neg P$	E$\Rightarrow$ 1,6
8	$\mid\ P$	E$\Rightarrow$ 2,4
9	$\neg L$	I$\neg$ 5,7,8

9.9 $\vdash P \Rightarrow P$

1	$\mid\ P$	H
2	$\mid\ P$	IT 1
3	$P \Rightarrow P$	I$\Rightarrow$ 1,2

9.10 $\vdash \neg(P \wedge \neg P)$

1	$\mid\ P \wedge \neg P$	H
2	$\mid\ P$	E$\wedge$ 1
3	$\mid\ \neg P$	E$\wedge$ 1
4	$\neg(P \wedge \neg P)$	I$\neg$ 1,2

9.11 $\vdash P \vee \neg P$

1	$\mid \neg(P \vee \neg P)$	H
2	$\mid \mid P$	H
3	$\mid \mid P \vee \neg P$	I$\vee$ 2
4	$\mid \mid \neg(P \vee \neg P)$	IT 1
5	$\mid \neg P$	I$\neg$ 2,3,4
6	$\mid P \vee \neg P$	I$\vee$ 5
7	$\mid \neg(P \vee \neg P)$	IT 1
8	$\neg\neg(P \vee \neg P)$	I$\neg$ 1,6,7
9	$P \vee \neg P$	E$\neg$ 8

9.12 $P \vee Q,\ \neg P \vdash Q$

1	$P \vee Q$	
2	$\neg P$	
3	$\mid P$	H
4	$\mid \mid \neg Q$	H
5	$\mid \mid \neg P$	IT 2
6	$\mid \mid P$	IT 3
7	$\mid \neg\neg Q$	I$\neg$ 4,5,6
8	$\mid Q$	E$\neg$ 7
9	$\mid Q$	H
10	$\mid Q$	IT 9
11	Q	E$\vee$ 1,8,10

9.13 $A \vee B,\ A \Rightarrow C,\ \neg D \Rightarrow \neg B \vdash C \vee D$

1	$A \vee B$	
2	$A \Rightarrow C$	
3	$\neg D \Rightarrow \neg B$	
4	$\mid A$	H
5	$\mid C$	E$\Rightarrow$ 2,4
6	$\mid C \vee D$	I$\vee$ 5
7	$\mid B$	H
8	$\mid \mid \neg D$	H
9	$\mid \mid \neg B$	E$\Rightarrow$ 3,8
10	$\mid \mid B$	IT 7
11	$\mid \neg\neg D$	I$\neg$ 8,9,10
12	$\mid D$	E$\neg$ 11
13	$\mid C \vee D$	I$\vee$ 12
14	$C \vee D$	E$\vee$ 1,6,13

9.14 $A \iff B \vdash (A \land B) \lor (\neg A \land \neg B)$

1	$(A \Rightarrow B) \land (B \Rightarrow A)$	
2	$\quad \neg(A \lor \neg A)$	H
3	$\quad\quad A$	H
4	$\quad\quad A \lor \neg A$	I$\lor$ 3
5	$\quad\quad \neg(A \lor \neg A)$	IT 2
6	$\quad \neg A$	I$\neg$ 3,4,5
7	$\quad A \lor \neg A$	I$\lor$ 6
8	$\quad \neg(A \lor \neg A)$	IT 2
9	$\neg\neg(A \lor \neg A)$	I$\neg$ 2,7,8
10	$A \lor \neg A$	E$\neg$ 9
11	$\quad A$	H
12	$\quad A \Rightarrow B$	E$\land$ 1
13	$\quad B$	E$\Rightarrow$ 12,11
14	$\quad A \land B$	I$\land$ 11,13
15	$\quad (A \land B) \lor (\neg A \land \neg B)$	I$\lor$ 14
16	$\quad \neg A$	H
17	$\quad\quad B$	H
18	$\quad\quad B \Rightarrow A$	E$\land$ 1
19	$\quad\quad A$	E$\Rightarrow$ 18,17
20	$\quad\quad \neg A$	IT 16
21	$\quad \neg B$	I$\neg$ 17,19,20
22	$\quad \neg A \land \neg B$	I$\land$ 16,21
23	$\quad (A \land B) \lor (\neg A \land \neg B)$	I$\lor$ 22
24	$(A \land B) \lor (\neg A \land \neg B)$	E$\lor$ 10,15,23

9.15 $P \vdash (P \Rightarrow Q) \Rightarrow Q$

1	P	
2	│ $P \Rightarrow Q$	H
3	│ Q	E$\Rightarrow$ 2,1
4	$(P \Rightarrow Q) \Rightarrow Q$	I$\Rightarrow$ 2,3

9.16 $P \Rightarrow Q \vdash (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$

1	$P \Rightarrow Q$	
2	│ $Q \Rightarrow R$	H
3	│ │ P	H
4	│ │ Q	E$\Rightarrow$ 1,3
5	│ │ R	E$\Rightarrow$ 2,4
6	│ $P \Rightarrow R$	I$\Rightarrow$ 3,5
7	$(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$	I$\Rightarrow$ 2,6

9.17 $P \Rightarrow Q,\ P \Rightarrow (Q \Rightarrow R) \vdash P \Rightarrow R$

1	$P \Rightarrow Q$	
2	$P \Rightarrow (Q \Rightarrow R)$	
3	│ P	H
4	│ Q	E$\Rightarrow$ 1,3
5	│ $Q \Rightarrow R$	E$\Rightarrow$ 2,3
6	│ R	E$\Rightarrow$ 5,4
7	$P \Rightarrow R$	I$\Rightarrow$ 3,6

9.18 $P \wedge Q \Rightarrow R \vdash P \Rightarrow (Q \Rightarrow R)$

1	$P \wedge Q \Rightarrow R$	
2	$\mid\ \underline{P}$	H
3	$\mid\ \mid\ \underline{Q}$	H
4	$\mid\ \mid\ P \wedge Q$	I$\wedge$ 2,3
5	$\mid\ \mid\ R$	E$\Rightarrow$ 1,4
6	$\mid\ Q \Rightarrow R$	I$\Rightarrow$ 3,5
7	$P \Rightarrow (Q \Rightarrow R)$	I$\Rightarrow$ 2,6

9.19 $\neg P \vdash P \Rightarrow Q$

1	$\neg P$	
2	$\mid\ \underline{P}$	H
3	$\mid\ \mid\ \underline{\neg Q}$	H
4	$\mid\ \mid\ \neg P$	IT 1
5	$\mid\ \mid\ P$	IT 2
6	$\mid\ \neg\neg Q$	I$\neg$ 3,4,5
7	$\mid\ Q$	E$\neg$ 6
8	$P \Rightarrow Q$	I$\Rightarrow$ 2,7

9.20 $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

1	$A \wedge (B \vee C)$	
2	A	E$\wedge$ 1
3	$B \vee C$	E$\wedge$ 1
4	$\mid\ \underline{B}$	H
5	$\mid\ A \wedge B$	I$\wedge$ 2,4
6	$\mid\ (A \wedge B) \vee (A \wedge C)$	I$\vee$ 5
7	$\mid\ \underline{C}$	H
8	$\mid\ A \wedge C$	I$\wedge$ 2,7
9	$\mid\ (A \wedge B) \vee (A \wedge C)$	I$\vee$ 8
10	$(A \wedge B) \vee (A \wedge C)$	E$\vee$ 3,6,9

9.21 $\neg A \vee B \vdash A \Rightarrow B$

1	$\neg A \vee B$	
2	│ A	H
3	│ │ $\neg A$	H
4	│ │ │ $\neg B$	H
5	│ │ │ A	IT 2
6	│ │ │ $\neg A$	IT 3
7	│ │ $\neg\neg B$	I$\neg$ 4,5,6
8	│ │ B	E$\neg$ 7
9	│ │ B	H
10	│ │ B	IT 9
11	│ B	E$\vee$ 1,8,10
12	$A \Rightarrow B$	I$\Rightarrow$ 2,11

9.22 $\vdash ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$

1	│ $(P \Rightarrow Q) \Rightarrow P$	H
2	│ │ $\neg P$	H
3	│ │ │ P	H
4	│ │ │ │ $\neg Q$	H
5	│ │ │ │ P	IT 3
6	│ │ │ │ $\neg P$	IT 2
7	│ │ │ $\neg\neg Q$	I$\neg$ 4,5,6
8	│ │ │ Q	E$\neg$ 7
9	│ │ $P \Rightarrow Q$	I$\Rightarrow$ 3,8
10	│ │ P	E$\Rightarrow$ 1,9
11	│ │ $\neg P$	IT 2
12	│ $\neg\neg P$	I$\neg$ 2,10,11
13	│ P	E$\neg$ 12
14	$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$	I$\Rightarrow$ 1,13

9.23 $Pa,\ Qa \vdash \exists x(Px \wedge Qx)$

1	Pa	
2	Qa	
3	$Pa \wedge Qa$	I$\wedge$ 1,2
4	$\exists x(Px \wedge Qx)$	I$\exists$ 3,a

9.24 $\forall x(Px \Rightarrow Qx),\ Pa \vdash Qa$

1	$\forall x(Px \Rightarrow Qx)$	
2	Pa	
3	$Pa \Rightarrow Qa$	E$\forall$ 1,a
4	Qa	E$\Rightarrow$ 3,2

9.25 $\forall x(Px \Rightarrow Qx),\ \forall x(Qx \Rightarrow Rx) \vdash \forall x(Px \Rightarrow Rx),$

1	$\forall x(Px \Rightarrow Qx)$	
2	$\forall x(Qx \Rightarrow Rx)$	
3	$\quad\mid Px$	H
4	$\quad\mid Px \Rightarrow Qx$	E$\forall$ 1,x
5	$\quad\mid Qx \Rightarrow Rx$	E$\forall$ 2,x
6	$\quad\mid Qx$	E$\Rightarrow$ 4,3
7	$\quad\mid Rx$	E$\Rightarrow$ 5,6
8	$Px \Rightarrow Rx$	I$\Rightarrow$ 3,7
9	$\forall x(Px \Rightarrow Rx)$	I$\forall$ 8

9.26 $\exists x\forall y Pxy \vdash \forall y\exists x Pxy$

1	$\exists x\forall y Pxy$	
2	$\quad\mid \forall y Pay$	H
3	$\quad\mid Pay$	E$\forall$ 2,y
4	$\quad\mid \exists x Pxy$	I$\exists$ 3,a
5	$\exists x Pxy$	E$\exists$ 1,2,4,a
6	$\forall y\exists x Pxy$	I$\forall$ 5